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THE FUNDAMENTAL SOLUTION METHOD FOR INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

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SUMMARY

A complete boundary integral formulation for incompressible Navier–Stokes equations with time discretization by operator splitting is developed using the fundamental solutions of the Helmholtz operator equation with different order. The numerical results for the lift and the drag hysteresis associated with a NACA0012 aerofoil oscillating in pitch show good agreement with available experimental data. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: fundamental solution method; integral equation method; Navier–Stokes equations

1. INTRODUCTION

A complete boundary integral formulation for inviscid flows governed by non-linear equations was developed previously [1] using the fundamental solutions of the Laplacian operator equation with different order. In this paper, the method presented in Reference [1] is extended further and a complete boundary integral formulation for incompressibe Navier–Stokes equations with time discretization by operator splitting is developed, using the fundamental solutions of the Helmholtz operator equation with different order. The numerical results for the lift and drag hysteresis associated with a NACA0012 aerofoil oscillating in pitch show good agreement with the available experimental data. The boundary integral formulation reduces the dimensionality of problems to be solved and the computational mesh to be generated is needed only on the boundaries of the domain. Thus, the required computer storage and computing time will be greatly reduced. Hence, it is an efficient method for solving Navier–Stokes equations.

2. THEORETICAL BASES

The non-dimensional incompressible Navier–Stokes equation are

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \dfrac{1}{Re}\nabla^2\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0, \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0, \quad \text{in } \Omega, \end{cases} \tag{1}$$

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where $\mathbf{u} = \{u_i\}$ is the flow velocity, p is the pressure and Re is the Reynold's number.

For simplicity, only Dirichlet conditions are considered, i.e.

$$\mathbf{u} = \mathbf{g} \ \text{ on } \ \Gamma, \quad \Gamma = \partial\Omega,$$

where $\int_\Gamma \mathbf{g} \cdot \mathbf{n} \, \mathrm{d}\Gamma = 0$, $\mathbf{n}$ is the unit vector of the outward normal at Γ. An initial condition must also be added, $\mathbf{u}(x, 0) = \mathbf{u}_0$. Using time discretization by operator splitting, the various operators occurring in the above governing equations are decoupled and the following Peaceman–Rachford schemes are obtained. For time step $n \geq 0$, assuming that $\mathbf{u}^n$ is known, $\mathbf{u}^{n+1/2}$, $p^{n+1/2}$ and $\mathbf{u}^{n+1}$ are computed by

$$\begin{cases} \dfrac{\mathbf{u}^{n+1/2} - \mathbf{u}^n}{\Delta t/2} - \dfrac{1}{2Re} \nabla^2 \mathbf{u}^{n+1/2} + \nabla p^{n+1/2} = \dfrac{1}{2Re} \nabla^2 \mathbf{u}^n - (\mathbf{u} \cdot \nabla)\mathbf{u}^n, \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u}^{n+1/2} = 0, \quad \text{in } \Omega, \\ \mathbf{u}^{n+1/2} = \mathbf{g}^{n+1/2}, \quad \text{on } \Gamma, \end{cases} \tag{2}$$

$$\begin{cases} \dfrac{\mathbf{u}^{n+1} - \mathbf{u}^{n+1/2}}{\Delta t/2} - \dfrac{1}{2Re} \nabla^2 \mathbf{u}^{n+1} + (\mathbf{u}^{n+1} \cdot \nabla)\mathbf{u}^{n+1} = \dfrac{1}{2Re} \nabla^2 \mathbf{u}^{n+1/2} - \nabla p^{n+1/2}, \quad \text{in } \Omega, \\ \mathbf{u}^{n+1} = \mathbf{g}^{n+1}, \quad \text{on } \Gamma, \end{cases} \tag{3}$$

where Δt is time interval.

Note that a linear variant of Equation (3) is obtained by substituting the linear term $(\mathbf{u}^{n+1/2} \cdot \nabla)\mathbf{u}^{n+1/2}$ to $(\mathbf{u}^{n+1} \cdot \nabla)\mathbf{u}^{n+1}$ in Equation (3) and $\mathbf{u}^{n+1}$ can be obtained by solving this linear variant Equation (3) iteratively. Under this condition, both Equation (2) and (3) are close to generalized Stokes problem as follows:

$$\begin{cases} \alpha \mathbf{u} - \nabla^2 \mathbf{u} + Re \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{4}$$

with $\alpha = Re/\Delta t$, $\mathbf{f} = (Re/\Delta t)\mathbf{u}^n - Re(\mathbf{u}^n \cdot \Delta)\mathbf{u}^n$. Hence, developing a complete boundary integral formulation for incompressible Navier–Stokes equations is now reduced to developing a complete boundary integral formulation for generalized Stokes problems.

Equation (4) can be solved using the conjugate gradient algorithm [2]. At each time step of iteration, p can be solved iteratively using the Laplace equation or Poisson equation, as shown in Reference [2], with the Dirichlet boundary condition prescribed by the well-known boundary integral method. A series of equations are then obtained:

$$\begin{cases} \alpha \mathbf{u} - \nabla^2 \mathbf{u} = \mathbf{f} - Re \nabla p = F, \quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}_g, \quad \text{on } \Gamma. \end{cases} \tag{5}$$

Multiplying Equation (5) with the fundamental solution H_0 of the Helmholtz operator equation with zero-order and then integrating it with respect to Ω gives

$$\int_\Omega (\alpha \mathbf{u} - \nabla^2 \mathbf{u}) H_0 \, \mathrm{d}\Omega = \int_\Omega F H_0 \, \mathrm{d}\Omega, \tag{6}$$

with $(\alpha - \nabla^2) H_0 = \delta(\mathbf{r})$. According to the Green theorem and the integrating property of impulse function δ, Equation (6) can be written as

$$c\mathbf{u}(\mathbf{r}) = \int_\Gamma \left(H_0 \frac{\partial \mathbf{u}}{\partial n} - \mathbf{u} \frac{\partial H_0}{\partial n} \right) \mathrm{d}\Gamma + \int_\Omega F H_0 \, \mathrm{d}\Omega. \tag{7}$$

For a smooth boundary, $c = 1/2$. In order to transform the domain integral in Equation (7) into a series of boundary integrals, two new functions, A_0 and H_1, are introduced, with $A_0 = F$, $(k^2 - \nabla^2)H_1 - H_0 = 0$, $k = \alpha^{1/2}$. Thus, the domain integral in Equation (7) can be expressed as

$$\int_\Omega F H_0 \, \mathrm{d}\Omega = \int_\Omega A_0 \square H_1 \, \mathrm{d}\Omega = \int_\Omega H_1 \square A_0 \, \mathrm{d}\Omega - \int_\Gamma \left(A_0 \frac{\partial H_1}{\partial n} - H_1 \frac{\partial A_0}{\partial n} \right) \mathrm{d}\Gamma, \tag{8}$$

where $\square = (k^2 - \nabla^2)$. Similarly, the domain integral in Equation (8) can be rewritten as

$$\int_\Omega H_1 \square A_0 \, \mathrm{d}\Omega = \int_\Gamma \left(A_1 \frac{\partial H_2}{\partial n} - H_2 \frac{\partial A_1}{\partial n} \right) \mathrm{d}\Gamma + \int_\Omega H_2 \square A_1 \, \mathrm{d}\Omega, \tag{9}$$

where $A_1 = \square A_0$, $\square H_2 - H_1 = 0$. The procedure can be generalized by introducing two sequence of functions defined by the recurrence formulae

$$A_{j+1} = \square A_j, \qquad \square H_{j+1} = H_j, \quad j = 0, 1, 2, \ldots$$

Thus the domain integral $\int_\Omega F H_0 \, \mathrm{d}\Omega$ can be expressed as the summations of infinite boundary integrals, i.e.

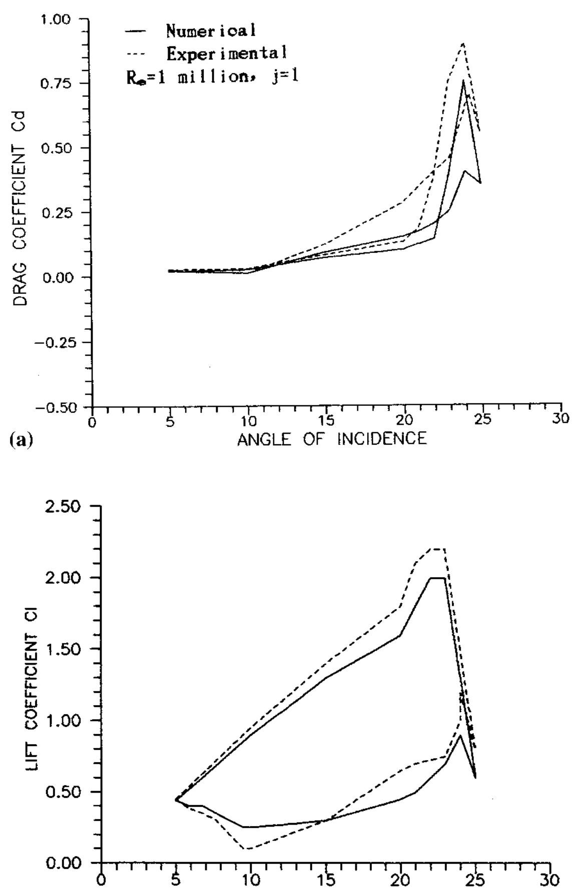

Figure 1. Lift and drag hysteresis for NACA0012 aerofoil oscillating in pitch.

$$\int_\Omega FH_0\,\mathrm{d}\Omega = \sum_{j=0}^{\infty}\int_\Gamma \left(A_j\frac{\partial H_{j+1}}{\partial n} - H_{j+1}\frac{\partial A_j}{\partial n}\right)\mathrm{d}\Gamma. \tag{10}$$

More generally, the jth order fundamental solution H_j satisfies

$$\Box H_j = H_{j-1}, \quad j = 1, 2, \ldots,$$

and can be expressed as

$$H_j = B_j(k\mathbf{r})K_j(k\mathbf{r}), \qquad B_0 = 1/2\pi, \qquad H_0 = B_0K_0(k\mathbf{r}),$$

where $B_j = B_{j-1}/2jk^2$ (for $j > 0$) and $K_j(x)$ represent the second type of modified Bessel function of jth order. Finally, substituting Equation (10) into (7), a complete boundary integral formulation for Equation (5) is obtained:

$$c\mathbf{u}(\mathbf{r}) = \int_\Gamma \left(H_0\frac{\partial \mathbf{u}}{\partial n} - \mathbf{u}\frac{\partial H_0}{\partial n}\right)\mathrm{d}\Gamma + \sum_{j=0}^{\infty}\int_\Gamma \left(A_j\frac{\partial H_{j+1}}{\partial n} - H_{j+1}\frac{\partial A_j}{\partial n}\right)\mathrm{d}\Gamma. \tag{11}$$

Notice that the introduction of factor jk^2 into the denominator of expression B_j guarantees the rapid convergence of Equation (11), especially for the flow with large Reynold's number.

3. NUMERICAL RESULTS

Figure 1 shows the lift and the drag hysteresis associated with a NACA0012 aerofoil oscillating in pitch about a pitch axis located on the aerofoil centerline at the quarter chord measured from the leading edge. The incidence angle for the oscillating aerofoil is varied with $\alpha = 5° + 20° \sin(\omega t)$, at a reduced frequency $\omega_R = 0.1$. The computing results show good agreement with available experimental data [3]. For large Reynold's number Re and small time interval Δt, the value of $k = \sqrt{\alpha} = \sqrt{Re/\Delta t}$ will be large enough to reduce the value of H_j greatly as j increases. In the present example ($Re = 10^6$, $\Delta t = 0.1$), the difference of results given by $j = 1$ and 2 already cannot be distinguished.

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